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LETTERS TO THE EDITOR

MECHANICAL SYSTEMS WITH A SINGLE VISCOUS DAMPER SUBJECT TO SEVERAL CONSTRAINT EQUATIONS

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1. INTRODUCTION

It is known that vibrations of a discrete linear mechanical system with n degrees of freedom are governed in the physical space by a matrix differential equation of order two. Assumed solutions of exponential form lead to an eigenvalue problem, the solution of which yields the eigencharacteristics of the mechanical system: i.e., eigenvalues and eigenvectors. Theoretically, the eigenvalues are obtained from the solvability condition of a set of n homogeneous linear equations, which in turn means that the corresponding determinant of coefficients has to vanish. This determinantal equation (characteristic equation) can for large n be solved numerically using a computer. In special cases, the characteristic equation can be manipulated further to be presented in the form of a simple analytical expression. One such special case is that of a linear system damped by means of a single viscous damper [1]. A previous study [2] was concerned with the above system in which a linear constraint relation between the system co-ordinates was also allowed. It was shown that the characteristic equation of this constrained system can also be reduced to a simple analytical expression. The present note is in some sense a generalization of the results in reference [2] because not only one but several constraint equations are allowed here.

2. THEORY

As is known, the motion of a linear discrete mechanical system with n degrees of freedom is governed in the physical space by the matrix differential equation of order two

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{D}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0}, \tag{1}$$

where $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{K}$ are the $n \times n$ mass, damping and stiffness matrices, respectively. $\mathbf{q}$ is the $n \times 1$ vector of the generalized co-ordinates. It will be assumed that the damping action on the system is due to only one viscous damper of damping constant c. The mathematical expression of this statement is

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$$\text{rank}\,\mathbf{D} = 1. \tag{2}$$

It is assumed further that the co-ordinates q_i of the system are subject to linear constraint equations of the form

$$\bar{\mathbf{a}}_p^{\mathrm{T}}\mathbf{q} = 0, \quad p = 1, \ldots, l, \tag{3}$$

where $\bar{\mathbf{a}}_p^{\mathrm{T}} = [\bar{a}_{1p}, \ldots, \bar{a}_{np}]$ and $\mathbf{q}^{\mathrm{T}} = [q_1, \ldots, q_n]$.

The main concern of the present note is to give the characteristic equation of the constrained system described above in the form of an analytical expression insofar as possible.

The transformation,

$$\mathbf{q} = \mathbf{\Phi}\boldsymbol{\eta}, \tag{4}$$

where $\mathbf{\Phi} = [\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_n]$ is the modal matrix of the undamped system, results in the following matrix differential equation in the modal space:

$$\ddot{\boldsymbol{\eta}} + \delta\mathbf{d}\mathbf{d}^{\mathrm{T}}\dot{\boldsymbol{\eta}} + \mathbf{\Omega}^2\boldsymbol{\eta} = \mathbf{0}, \tag{5}$$

where $\boldsymbol{\eta}^{\mathrm{T}} = [\eta_1, \ldots, \eta_n]$. The relations

$$\mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{\Phi} = \mathbf{I}, \quad \text{and} \quad \mathbf{\Phi}^{\mathrm{T}}\mathbf{K}\mathbf{\Phi} = \mathbf{\Omega}^2 = \mathbf{diag}(\omega_i^2), \quad i = 1, \ldots, n, \tag{6}$$

are used, which are due to the mass orthonormalization of the mode vectors $\mathbf{\Phi}_i$. $\mathbf{I}$ denotes the $n \times n$ unit matrix.

It is worth noting that the transformed damping matrix $\mathbf{\Phi}^{\mathrm{T}}\mathbf{D}\mathbf{\Phi}$ can be written in a dyadic form as $\delta\mathbf{d}\mathbf{d}^{\mathrm{T}}$. The coupling of the eigenmodes is affected by one viscous damper which acts on the vibrating system via the normalized orientation vector $\mathbf{d}$ with $\mathbf{d}^{\mathrm{T}}\mathbf{d} = 1$. The magnitude of the damping is represented by the damping constant δ [2, 3]. The matrix $\mathbf{\Omega}^2$ in equations (6) is defined as the diagonal matrix of the squares of the eigenfrequencies of the undamped mechanical system.

The transformed constraint equations take the form

$$\mathbf{a}_p^{\mathrm{T}}\boldsymbol{\eta} = 0, \quad p = 1, \ldots, l, \tag{7}$$

with $\mathbf{a}_p^{\mathrm{T}} = [a_{1p}, \ldots, a_{np}]$, where $a_{ip} = \bar{\mathbf{a}}_p^{\mathrm{T}}\mathbf{\Phi}_i$.

By means of the Lagrange's equations formalism in connection with Lagrange multipliers, equations (5) and (7) can be combined as

$$\ddot{\boldsymbol{\eta}} + \delta\mathbf{d}\mathbf{d}^{\mathrm{T}}\dot{\boldsymbol{\eta}} + \mathbf{\Omega}^2\eta = \sum_{p=1}^{l}\mu_p\mathbf{a}_p, \tag{8}$$

where μ_p denotes the corresponding Lagrange multiplier.

If exponential solutions of the form

$$\boldsymbol{\eta} = \boldsymbol{\alpha}\,\mathrm{e}^{\lambda t}, \quad \mu_p = \beta_p\,\mathrm{e}^{\lambda t}, \quad p = 1, \ldots, l, \tag{9}$$

are assumed and these are substituted into equation (8), where λ represents the unknown eigenvalue of the constrained system and $\boldsymbol{\alpha}$ and β_p are unknown amplitudes,

$$\boldsymbol{\alpha} = \sum_{p=1}^{l} \beta_p (\lambda^2 \mathbf{I} + \lambda\delta \mathbf{d}\mathbf{d}^{\mathrm{T}} + \boldsymbol{\Omega}^2)^{-1} \mathbf{a}_p \tag{10}$$

is obtained. Then, after substitution into the constraint equation (7), the following equations are obtained:

$$[\mathbf{a}_p^{\mathrm{T}}(\lambda^2\mathbf{I} + \boldsymbol{\Omega}^2 + \lambda\delta\mathbf{d}\mathbf{d}^{\mathrm{T}})^{-1}\mathbf{a}_1]\beta_1 + \cdots + [\mathbf{a}_p^{\mathrm{T}}(\lambda^2\mathbf{I} + \boldsymbol{\Omega}^2 + \lambda\delta\mathbf{d}\mathbf{d}^{\mathrm{T}})^{-1}\mathbf{a}_l]\beta_l = 0, \qquad p = 1, \ldots, l. \tag{11}$$

Equating to zero the determinant Δ of the coefficient matrix of this set of homogeneous equations for β_p results in the characteristic equation of the system

$$\Delta(\lambda) = 0, \tag{12}$$

where the p, qth element of the $l \times l$ determinant Δ is defined as

$$\Delta_{pq} = \mathbf{a}_p^{\mathrm{T}}(\lambda^2\mathbf{I} + \boldsymbol{\Omega}^2 + \lambda\delta\mathbf{d}\mathbf{d}^{\mathrm{T}})^{-1}\mathbf{a}_q. \tag{13}$$

In order to evaluate Δ_{pq} analytically, first one has to evaluate the matrix $(\lambda^2\mathbf{I} + \boldsymbol{\Omega}^2 + \lambda\delta\mathbf{d}\mathbf{d}^{\mathrm{T}})^{-1}$. To this end, a matrix inversion formula can be used from matrix theory which gives the inverse of the sum of a nonsingular square matrix and a dyadic [4]:

$$(\mathbf{A} + \mathbf{u}\mathbf{v}^{\mathrm{T}})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}/(1 + \mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{u}). \tag{14}$$

The speciality of this formula is that if $\mathbf{A}^{-1}$ is known, the inverse of $\mathbf{A}$ augmented by a rank-one matrix can be obtained by a simple modification of the known $\mathbf{A}^{-1}$.

By identifying $\lambda^2\mathbf{I} + \boldsymbol{\Omega}^2 = \mathbf{A}$, $\delta\lambda\mathbf{d} = \mathbf{u}$, $\mathbf{d} = \mathbf{v}$ the following equation can be written:

$$(\lambda^2\mathbf{I} + \boldsymbol{\Omega}^2 + \lambda\delta\mathbf{d}\mathbf{d}^{\mathrm{T}})^{-1} = (\lambda^2\mathbf{I} + \boldsymbol{\Omega}^2)^{-1} - \frac{(\lambda^2\mathbf{I} + \boldsymbol{\Omega}^2)^{-1}\delta\lambda\mathbf{d}\mathbf{d}^{\mathrm{T}}(\lambda^2\mathbf{I} + \boldsymbol{\Omega}^2)^{-1}}{1 + \delta\lambda\mathbf{d}^{\mathrm{T}}(\lambda^2\mathbf{I} + \boldsymbol{\Omega}^2)^{-1}\mathbf{d}}. \tag{15}$$

When the denominator on the right side is equated to zero, nothing else but the characteristic equation of the damped system without any constraint will be obtained [1]. This will be denoted by $P(\lambda)$, where,

$$P(\lambda) = 1 + \lambda\delta \sum_{k=1}^{n} \frac{d_k^2}{\lambda^2 + \omega_k^2}. \tag{16}$$

$P(\lambda) = 0$ results in a polynomial equation of order $2n$ in λ for the eigenvalues of the unconstrained system. Hence, the expression (13) takes the form

$$\Delta_{pq} = \frac{1}{P(\lambda)}[P(\lambda)\mathbf{a}_p^{\mathrm{T}}(\lambda^2\mathbf{I}+\boldsymbol{\Omega}^2)^{-1}\mathbf{a}_q - \delta\lambda\mathbf{a}_p^{\mathrm{T}}[(\lambda^2\mathbf{I}+\boldsymbol{\Omega}^2)^{-1}\mathbf{d}][(\lambda^2\mathbf{I}+\boldsymbol{\Omega}^2)^{-1}\mathbf{d}]^{\mathrm{T}}\mathbf{a}_q]. \tag{17}$$

After some manipulations, it can be shown that

$$\mathbf{a}_p^{\mathrm{T}}(\lambda^2\mathbf{I}+\boldsymbol{\Omega}^2)^{-1}\mathbf{a}_q = \sum_{i=1}^{n}\frac{a_{ip}a_{iq}}{\lambda^2+\omega_i^2},$$

$$\mathbf{a}_p^{\mathrm{T}}[\lambda^2\mathbf{I}+\boldsymbol{\Omega}^2)^{-1}\mathbf{d}][(\lambda^2\mathbf{I}+\boldsymbol{\Omega}^2)^{-1}\mathbf{d}]^{\mathrm{T}}\mathbf{a}_q = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{a_{iq}a_{jp}d_id_j}{(\lambda^2+\omega_i^2)(\lambda^2+\omega_j^2)}. \tag{18}$$

If expressions (16) and (18) are substituted into equation (17)

$$\Delta_{pq} = \frac{1}{P(\lambda)}\left[\sum_{i=1}^{n}\frac{a_{ip}a_{iq}}{\lambda^2+\omega_i^2} + \lambda\delta\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{d_ia_{jp}(d_ia_{jq}-d_ja_{iq})}{(\lambda^2+\omega_i^2)(\lambda^2+\omega_j^2)}\right], \quad i \neq j, \tag{19}$$

is obtained. After arranging the terms in the double sum, expression (19) can also be expressed in a more symmetrical form as

$$\Delta_{pq} = \frac{1}{P(\lambda)}\left[\sum_{i=1}^{n}\frac{a_{ip}a_{iq}}{\lambda^2+\omega_i^2} + \lambda\delta\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{(d_ia_{jp}-d_ja_{ip})(d_ia_{jq}-d_ja_{iq})}{(\lambda^2+\omega_i^2)(\lambda^2+\omega_j^2)}\right]. \tag{20}$$

The special case $\delta = 0$ corresponds to the undamped system in which case equation (20) reduces to the simple form

$$\Delta_{pq} = \frac{1}{P(\lambda)}\sum_{i=1}^{n}\frac{a_{ip}a_{iq}}{\lambda^2+\omega_i^2}. \tag{21}$$

Upon recognizing that the factor $1/P(\lambda)$ appears in all elements Δ_{pq} of the determinant Δ, it can be stated that the characteristic equation of the constrained system under investigation can be obtained simply by setting the determinant of the $l \times l$ matrix $\mathbf{R}$ equal to zero,

$$\det \mathbf{R} = 0, \tag{22}$$

the p, qth element of which is defined as

$$r_{pq} = \sum_{i=1}^{n}\frac{a_{ip}a_{iq}}{\lambda^2+\omega_i^2} + \lambda\delta\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{(d_ia_{jp}-d_ja_{ip})(d_ia_{jq}-d_ja_{iq})}{(\lambda^2+\omega_i^2)(\lambda^2+\omega_j^2)}. \tag{23}$$

On the other hand, the eigenvectors of the constrained system are given by equation (10) where the coefficients β_p are obtained from the solution of the set of homogeneous equations given in equation (11).

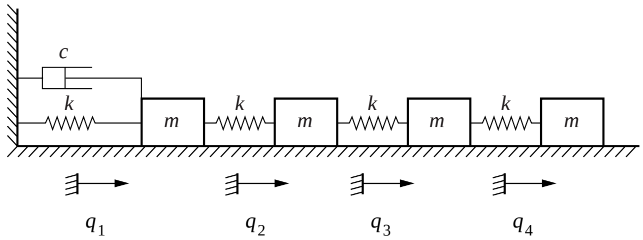

Figure 1. Unconstrained sample system with four degrees of freedom.

3. NUMERICAL APPLICATIONS

3.1. *Example* 1

Although the theoretical results of the preceding section are quite general, the simple system shown in Figure 1 is taken as a first illustrative example. This example will aid in obtaining a better physical insight into the results that are obtained.

It is a simple matter to show that the eigenfrequencies of the undamped oscillator in Figure 1 are $\omega_1 = 0{\cdot}347296355\omega_0$, $\omega_2 = \omega_0$, $\omega_3 = 1{\cdot}532088886\omega_0$, $\omega_4 = 1{\cdot}879385242\omega_0$, where $\omega_0^2 = k/m$.

The corresponding modal matrix is as follows

$$\mathbf{\Phi} = [\mathbf{\Phi}_1\mathbf{\Phi}_2\mathbf{\Phi}_3\mathbf{\Phi}_4]$$

$$= \frac{2}{3\sqrt{m}}\begin{bmatrix} 0{\cdot}342020143 & 0{\cdot}866025404 & 0{\cdot}984807753 & 0{\cdot}642787610 \\ 0{\cdot}642787610 & 0{\cdot}866025404 & -0{\cdot}342020143 & -0{\cdot}984807753 \\ 0{\cdot}866025404 & 0 & -0{\cdot}866025404 & 0{\cdot}866025404 \\ 0{\cdot}984807753 & -0{\cdot}866025404 & 0{\cdot}642787610 & -0{\cdot}342020143 \end{bmatrix}.$$

δ and $\mathbf{d}$ in equation (5) are obtained as $\delta = c/m$ and $\mathbf{d} = [d_1, \ldots, d_4]^{\mathrm{T}} = [0{\cdot}228013429\ 0{\cdot}577350269\ 0{\cdot}656538502\ 0{\cdot}428525073]^{\mathrm{T}}$.

Now, let it be assumed that two constraints of the form $q_2 = \alpha q_1$, $q_4 = \beta q_3$ are imposed on the system such that according to equation (3)

$$\bar{\mathbf{a}}_1 = [-\alpha\ 1\ 0\ 0]^{\mathrm{T}}, \quad \bar{\mathbf{a}}_2 = [0\ 0\ -\beta\ 1]^{\mathrm{T}}$$

are obtained. These determine $\mathbf{a}_1$ and $\mathbf{a}_2$ in equation (7) as

$$\mathbf{a}_1 = \begin{bmatrix} -0{\cdot}228013429\alpha + 0{\cdot}428525073 \\ 0{\cdot}577350269(1-\alpha) \\ -0{\cdot}656538502\alpha - 0{\cdot}228013429 \\ -0{\cdot}428525073\alpha - 0{\cdot}656538502 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} -0{\cdot}577350269\beta + 0{\cdot}656538502 \\ -0{\cdot}577350269 \\ 0{\cdot}577350269\beta + 0{\cdot}428525073 \\ -0{\cdot}577350269\beta - 0{\cdot}228013429 \end{bmatrix}.$$

The constrained system is shown in Figure 2, for $\alpha = \beta = 1$. After choosing further $k = 100$, $m = 5$, $c = 20$, the eigenvalues of this special system are given in Table 1. The complex numbers in the first column are the eigenvalues obtained by solving the eigenvalue problem of the system in Figure 2. The numbers in the

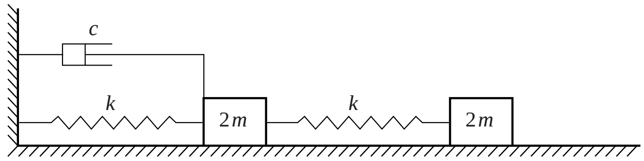

Figure 2. Constrained system, obtained from the system in Figure 1 by taking $q_1 = q_2$, $q_3 = q_4$.

second column are obtained by solving equation (22) numerically with MATLAB. The agreement of the complex numbers in both columns is excellent.

In order to gain insight on how accurately the eigenvectors can be obtained, the eigenvectors of the system in Figure 2 are given in Table 2 according to the representation $\tilde{\mathbf{x}}_j^{\mathrm{T}} = [\tilde{\mathbf{y}}_j^{\mathrm{T}} \mid \lambda_j \tilde{\mathbf{y}}_j^{\mathrm{T}}]$. The eigenvectors in the first column are obtained directly by solving the eigenvalue problem of the mechanical system shown in Figure 2. The eigenvectors in the second column are determined using (10), (4) and (9). The agreement is again very good.

3.2. *Example* 2

As a second example the continuous system shown in Figure 3 is considered. Here, the unconstrained system consists of a clamped–free Bernoulli–Euler beam which is damped at $x = l$ by a viscous damper of constant c. Bending rigidity, length and mass per unit length of the beam are EI, L and m respectively. The equation of motion of small vibrations of the beam is

$$EIw^{\mathrm{IV}}(x, t) + m\ddot{w}(x, t) + c\dot{w}(x, t)\delta(x - l) = 0, \tag{24}$$

where $w(x, t)$ denotes the bending displacement at point x and time t and $\delta(x)$ is the Dirac function. The primes and overdots denote partial derivatives with respect to x and t, respectively.

The corresponding boundary conditions are

$$w(0, t) = w'(0, t) = w''(L, t) = w'''(L, t) = 0. \tag{25}$$

Substitution of a solution of the form

$$w(x, t) = \sum_{k=1}^{n} w_k(x)\eta_k(t), \tag{26}$$

where $w_k(x)$ are mass-orthonormalized eigenfunctions of a clamped–free beam and $\eta_k(t)$ are generalized coordinates into equation (24) and application of Galerkin's procedure yields, after some arrangement, the equations of motion in

Table 1

Eigenvalues of the system in Figure 2 with $k = 100$, $m = 5$ and $c = 20$

$\lambda_{1,2}$	$-0{\cdot}283914 \pm 1{\cdot}970835$i	$-0{\cdot}283914 \pm 1{\cdot}970835$i
$\lambda_{3,4}$	$-0{\cdot}716086 \pm 4{\cdot}970835$i	$-0{\cdot}716128 \pm 4{\cdot}970853$i

TABLE 2
Eigenvectors of the system in Figure 2

$\tilde{\mathbf{y}}_{1,2}$	$\begin{bmatrix} 0{\cdot}619642 \pm 0{\cdot}111909i \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0{\cdot}619642 \pm 0{\cdot}111909i \\ 1 \end{bmatrix}$
$\tilde{\mathbf{y}}_{3,4}$	$\begin{bmatrix} -1{\cdot}419642 \pm 0{\cdot}711909\mathrm{i} \\ 1 \end{bmatrix}$	$\begin{bmatrix} -1{\cdot}419693 \pm 0{\cdot}711806\mathrm{i} \\ 1 \end{bmatrix}$

the modal space just as in the form of equation (5), with

$$\omega_0^2 = EI/mL^4, \quad \omega_i^2 = \bar{\beta}_i^4 \omega_0^2, \quad \bar{\beta}_1 = 1{\cdot}875104, \quad \bar{\beta}_2 = 4{\cdot}694091, \ldots,$$

$$\bar{l} = l/L, \quad \delta = c\sum_{k=1}^{n} w_k^2(\bar{l}), \quad \mathbf{d} = [d_1, \ldots, d_n]^{\mathrm{T}}, \quad d_k = w_k(\bar{l}) \Bigg/ \sqrt{\sum_{i=1}^{n} w_i^2(\bar{l})},$$

$$w_k(\bar{l}) = \frac{1}{\sqrt{mL}}[\mathrm{ch}\,\bar{\beta}_k\bar{l} - \cos\bar{\beta}_k\bar{l} - \bar{\eta}_k(\mathrm{sh}\,\bar{\beta}_k\bar{l} - \sin\bar{\beta}_k\bar{l})],$$

$$\bar{\eta}_k = (\mathrm{ch}\,\bar{\beta}_k + \cos\bar{\beta}_k)/(\mathrm{sh}\,\bar{\beta}_k + \sin\bar{\beta}_k). \tag{27}$$

It is now assumed that the vibrating beam in Figure 3 has to carry at the tip a heavy mass M with rotational inertia J, as depicted by the dashed lines. The characteristic equation of the so-constrained system can be obtained now from equation (22) directly.

The fourth boundary condition has to be modified as

$$EIw'''(L, t) - M\ddot{w}(L, t) = 0, \tag{28}$$

which represents the force balance at the free end leading to a relation of the type (7) between the modal co-ordinates $\eta_i(t)$, where

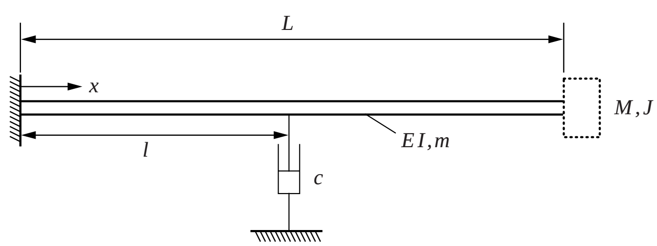

Figure 3. Viscously damped cantilever beam (unconstrained system) which is carrying a heavy tip mass M with rotary inertia J.

TABLE 3

Eigenvalues of the system in Figure 3

From $\bar{\mathbf{A}}$ in equation (32)	From equation (22)
$-0{\cdot}001493 \pm 5{\cdot}311867\mathrm{i}$	$-0{\cdot}001490 \pm 5{\cdot}299738\mathrm{i}$
$-0{\cdot}017708 \pm 20{\cdot}651277\mathrm{i}$	$-0{\cdot}017700 \pm 20{\cdot}650928\mathrm{i}$
$-1{\cdot}752293 \pm 184{\cdot}673144\mathrm{i}$	$-1{\cdot}750058 \pm 184{\cdot}667724\mathrm{i}$
$-6{\cdot}170935 \pm 500{\cdot}877648\mathrm{i}$	$-6{\cdot}167898 \pm 495{\cdot}727023\mathrm{i}$
$-9{\cdot}016424 \pm 979{\cdot}854106\mathrm{i}$	$-9{\cdot}044294 \pm 974{\cdot}939716\mathrm{i}$

$$a_{i1} = \bar{\beta}_i^3[\mathrm{sh}\,\bar{\beta}_i - \sin\bar{\beta}_i - \bar{\eta}_i(\mathrm{ch}\,\bar{\beta}_i + \cos\bar{\beta}_i)] - \frac{\alpha_M\lambda^2}{\omega_0^2}[\mathrm{ch}\,\bar{\beta}_i - \cos\bar{\beta}_i - \bar{\eta}_i(\mathrm{sh}\,\bar{\beta}_i - \sin\bar{\beta}_i)], \quad i = 1, \ldots, n, \tag{29}$$

with $\alpha_M = M/mL$.

The third boundary condition is now [5]

$$EIw''(L, t) - J\ddot{w}'(L, t) = 0, \tag{30}$$

which represents the moment balance at the free end. This leads to a relation of the type (7) between the modal co-ordinates $\eta_i(t)$, where

$$a_{i2} = \bar{\beta}_i^2[\mathrm{ch}\,\bar{\beta}_i + \cos\bar{\beta}_i - \bar{\eta}_i(\mathrm{sh}\,\bar{\beta}_i + \sin\bar{\beta}_i)] + \frac{\bar{J}\lambda^2\bar{\beta}_i}{\omega_0^2}[\mathrm{sh}\,\bar{\beta}_i + \sin\bar{\beta}_i - \bar{\eta}_i(\mathrm{ch}\,\bar{\beta}_i - \cos\bar{\beta}_i)], \tag{31}$$

with $\bar{J} = J/mL^3$, J being the rotational inertia of the tip mass.

In order to validate equation (22) numerically, one must be able to obtain the eigenvalues of the system in Figure 3 in another way as well. By using the results of reference [6], it is easy to show that the dimensionless eigenvalues $\lambda^* = \lambda/\omega_0$ can be obtained as the eigenvalues of the $2n \times 2n$ matrix

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{*-1}\mathbf{K}^* & -\mathbf{M}^{*-1}\mathbf{D}^* \end{bmatrix}, \tag{32}$$

with

$$\mathbf{M}^* = \mathbf{I} + \alpha_M\mathbf{a}^*(1)\mathbf{a}^{*\mathrm{T}}(1) + \bar{J}\mathbf{a}^{*\prime}(1)\mathbf{a}^{*\prime\mathrm{T}}(1), \quad \mathbf{D}^* = \bar{c}\mathbf{a}^*(\bar{l})\mathbf{a}^{*\mathrm{T}}(\bar{l}), \quad \bar{\mathbf{K}} = \mathbf{B}, \tag{33}$$

where

$$\mathbf{B} = \mathrm{diag}(\bar{\beta}_i^4), \quad \bar{c} = c/mL\omega_0, \quad \bar{x} = x/L, \quad \mathbf{a}^*(\bar{x}) = [a_1^*(\bar{x}), \ldots, a_n^*(\bar{x})]^{\mathrm{T}},$$

$$a_k^*(\bar{x}) = \mathrm{ch}\,\bar{\beta}_k\bar{x} - \cos\bar{\beta}_k\bar{x} - \bar{\eta}_k(\mathrm{sh}\,\bar{\beta}_k\bar{x} - \sin\bar{\beta}_k\bar{x}), \quad \mathbf{a}^{*\prime}(\bar{x}) = \mathrm{d}\mathbf{a}^*(\bar{x})/\mathrm{d}\bar{x}, \tag{34}$$

$\mathbf{I}$, $\mathbf{0}$ are $n \times n$ unit and zero matrix respectively. $\bar{\eta}_k$ is given in equations (27) and α_M in connection with equation (29).

The following numerical values are chosen for the physical data of the vibration system in Figure 3: $E = 7 \times 10^{10}$ N/m^2, $I = (0{\cdot}05 \times 0{\cdot}005^3)/12$ m^4, $L = 1$ m, $mL = 0{\cdot}675$ kg, $\bar{l} = l/L = 0{\cdot}2$, $c = 5$ N/m/s, $J = 0{\cdot}675$ kgm^2.

The first five pairs of eigenvalues λ of this system are recorded in Table 3. The complex numbers in the first column are results of matrix $\bar{\mathbf{A}}$ in equation (32) which were obtained by MATLAB. The complex numbers in the second column are values, obtained as the roots of equation (22) by MATLAB. In both cases $n = 10$ is assumed. Inspection of the complex numbers in both columns indicates clearly that their agreement is very good.

4. CONCLUSIONS

This study has dealt with a linear discrete mechanical system which is damped by a single viscous damper. The co-ordinates of the system are assumed to be subject to several linear constraint relations. By using a matrix inversion formula from matrix theory, analytical expressions for the elements of the characteristic determinant of the constrained system are obtained.

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